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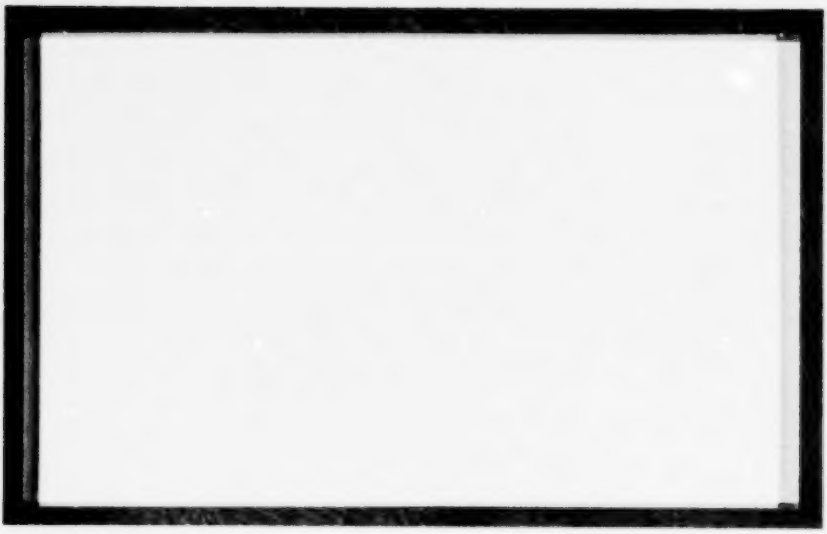

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COST ALLOCATION: THE TRAVELING SALESMAN, BINPACKING, AND THE KNAPSACK

by

Moshe Dror*

* INRS-Telecommunications, 3 Place du commerce, Verdun, Québec, H3E 1H6, and Centre de recherche sur les transports, Université de Montréal, C.P. 6128, Succursale A, Montréal, Québec, H3C 3J7.

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ABSTRACT

We examine the problem of cost allocation for the traveling salesman problem - (TSP). We note the significance of a 'home' city in the TSP for designing cost allocation schemes focusing on TSP's with triangular inequality. In addition, a similar cost allocation question is stated for the binpacking and the knapsack problems.

Key words : traveling salesman problem, cost allocation, cost game.

RÉSUMÉ

On considère la répartition des coûts entre les clients dans le problème du voyageur de commerce. L'importance du dépôt est soulignée pour les cas où les coûts satisfont l'inégalité du triangle. Une question est également examinée pour les problèmes dits de "binpacking" et du "knapsack".

Mots-clés : problème du voyageur de commerce, répartition des coûts.

1. INTRODUCTION

In the opening chapter, (first sentence) of the book titled **The Traveling Salesman Problem**, (Edited by Lawler, Lenstra, Rinnooy Kan, and Shmoys, John Wiley & Sons Ltd. 1985), Hoffman and Wolfe introduce the Traveling Salesman Problem - (TSP) as follows: "If a salesman, starting from his home city, is to visit exactly once each city on a given list and then return home, it is plausible for him to select the order in which he visits the cities so that the total of the distances traveled in his tour is as small as possible". On the other hand, in graph theory terminology the TSP is defined as the problem of finding a minimum-weight Hamilton cycle in a weighted complete graph (there is no mention of 'home'). In the book **Computers and Intractability**, by Garey and Johnson, W.H. Freeman and Company, San Francisco, 1979, the TSP is described as minimization of the expression $\left[\sum_{i=1}^{m-1} d(c_{\pi(i)}, c_{\pi(i+1)})\right] + d(c_{\pi(m)}, c_{\pi(1)})$, over all permutations π of $\{1, 2, ..., m\}$, where $C = \{c_1, c_2, ..., c_m\}$ is a finite set of "cities", for each pair c_i, c_j we have $d(c_i, c_j)$ as the distance between those cities, and a solution is an ordering $< c_{\pi(1)}, c_{\pi(2)}, ..., c_{\pi(m)} >$. Another well known mathematical description of the TSP due to Miller, Tucker, and Zemlin (1960), is given by:

Find variables x_{ij} and u_i for i,j = 1,2, ..., m that minimize

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{m} d(c_i, c_j) x_{ij} \qquad (1)$$

subject to:

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad \text{j} = 1,2, ..., \text{m}, \qquad (2)$$

$$\sum_{j=1}^{m} x_{ij} = 1 \qquad \text{i} = 1,2, ..., \text{m}, \qquad (3)$$

$$u_i - u_j + m x_{ij} \leq m - 1 \qquad \text{i,j} = 2,3, ..., \text{m } i \neq j \qquad (4)$$

$$x_{ij} = 0 \text{ or } 1 \qquad \text{for all i,j} \qquad (5)$$

In a verbal description of the TSP, the temptation of designating one "city" on a list as the "home city" is to great to resist. In the graph theoretical description, or the other two mathematical descriptions of the TSP, there is no "home city". It is irrelevant for a solution to the TSP whether the cities are described as home and customers, nodes on a graph, or just a finite set $C = \{c_1, c_2, ..., c_m\}$. When attempting to examine the TSP in the context of cost allocation within the frame of a cost game, the existence - distinction of a "home city", or the luck of such distinction is crucial.

In the cost game we have a pair $< C, Z >$, where the finite set C denotes the set of participants, each of them indexed by a number. The subsets of C are called coalitions. The 'cost' function $Z : 2^C \rightarrow$ R assigns to any nonempty coalition S the **minimal** cost Z(S) incurred if the set S is 'served' together in isolation of the other members in $C \setminus S$. For the empty coalition $Z(\emptyset) = 0$.

Already at this junction a basic problem crops up. According to Tijs and Driessen (1986), the interpretation of the cost function Z implies that Z is subadditive, i.e., $Z(S \cup T) \leq Z(S) + Z(T)$ for all $S, T \subset C$ with $S \cap T = \emptyset$. For the TSP as defined for example by (1) - (5), this subadditivity of the cost function is false in general unless the TSP is reexamined and modified. It can be illustrated by a simple symmetric 4 point problem: $C = \{c_1, c_2, c_3, c_4\}$. $d(c_1, c_2) = d(c_3, c_4) = 10, d(c_2, c_3) = d(c_4, c_1) = 1$. The rest of the distances are very long. The optimal TSP solution has the length of 22. If $S = \{c_2, c_3\}$ and $T = \{c_1, c_4\}$, the Z(S) = Z(T) = 2 and obviously $22 \not\leq 4$. This example illustrates that the TSP as defined without the designation of a "home city" and modification of its description is highly problematic from the cost game prospective since the TSP cost game is not subadditive in general.

A cost allocation method is a function $f :< C, Z > \rightarrow \mathrm{R}^{\mathrm{m}}$, defined for all cost games $< C, Z >$ such that $f(< C, Z >) = (a_1, a_2, ..., a_m)$ and $\sum_{i=1}^{m} a_i = Z(C)$, where a_i is the charge assessed to c_i.

The cost allocation problem is the problem of finding the best cost allocation method.

By skimming through the vast literature on this topic one realizes very quickly that no final answer to the cost allocation problem should be expected from the theoretical investigation. Best cost allocation might be a negotiated concept of the participants with respect to power, fairness, and many other factors. In this paper we examine the cost allocation issue for a number of combinatorial optimization problems. The main moto being the TSP.

The question of interest here is: Given an optimal solution to a TSP, what portion of the total distance should/could be attributed to a particular city ? Setting mathematical curiosity aside, this question is asked for example in a real life problem of designing efficient distribution system for home heating oil. (See Dror and Ball, 1987.)

2. SOME PROPERTIES OF THE TSP COST GAME

In the cost allocation literature an often cited example of a cost allocation problem with descriptive similarity to the TSP is that of allocating travel costs along a fixed itinerary of cities. (See for example Young, 1985 p. 7, Fishburn and Pollak (1983), and Shapley (1981).) The statement of the problem in the paper by Fishburn and Pollak is as follows: H.O. Pollak "flew from Newark, New Jersey, to Columbus, Ohio, to Raleigh/Durham to Tempa to Washington, D.C., then back to Newark. He went to Columbus on Bell Laboratories business, to Raleigh/Durham for the State of North Carolina, to Tempa for personal reasons, and to Washington on National Science Foundation business. After returning home, he had to decide how much of his air fare to charge to each sponsor of his trip, namely Bell Labs, the State of North Carolina, himself, and the NSF. He wished to do this in as fair a manner as possible."

The subadditvity of the cost function in the case of fixed itinerary is achived simply through the option of flying home (or other intermidiate stop) whenever the cost of a direct flight between two cities exceeds the cost of returning home first and then continueing to the next destination. If this option is not taken into account, then the cost function is not necessary subadditive since the airline pricing policies are based not on distance alone but on demand, competition, etc. There is no attempt of distance optimization in the fixed itinerary case and the similarity to the TSP is more perceived than real. Nevertheless, it is of interest to examine what Fishburn and Pollak, 1983, refer to as the "Fairness Criteria" in case of the fixed itinerary and later relate the TSP cost game to these criteria.

A1. $\sum_{i=1}^{m} a_i = Z(C)$

The allocated costs equal the total cost without subsidizing the trip or profiting from it. (It is also known as the *efficiency* property, [.].)

A2. $a_i \geq 0$ for all i

This criteria implies that no cities receive 'kickbacks' at the expanse of other cities.

A3. $a_i \leq 2d(c_1, c_i)$ for all $i \neq 1$

In the fixed itinerary case this condition implies that it would be unfair to charge a city c_i more than the round-trip cost between home and city i. If the distance matrix $\{d(c_i, c_j)\}$ satisfies the triangle inequality than regardless which city in the set C is the home city (city c_1), $Z(C) \leq 2\sum_{i=2}^{m} d(c_1, c_i)$, and **A1, A2, A3** are simulteneously feasible.

A4. $a_i = a_{\pi(i)}$ for all permutations π of $\{1, 2, ..., m\}$

This criteria implies that the naming (numbering) of cities is not relevant to their cost allocation. It is called the *anonymity/neutrality* condition in Foshburn and Pollak, 1983, and *anonymity* property in Young, 1985.

A5. If the matrix $\{d'(c_i, c_j)\} \geq \{d(c_i, c_j)\}$, then $a'_i \geq a_i$ for all i.

($\{d'(c_i, c_j)\} \geq \{d(c_i, c_j)\}$ implies that $d'(c_i, c_j) \geq d(c_i, c_j)$ for all i,j)

This condition is called the *monotonicity* condition. It implies that if the overall cost increases then there cannot be a decrease in the cost allocated to any specific city.

A6. If $d(c_i, c_j) = d'(c_i, c_j) + d''(c_i, c_j)$ then $a_i = a'_i + a''_i$

If the cost matrix $\{d(c_i, c_j)\}$ is seperable into two distinct and independent cost components, then the cost allocated to a city c_i is also seperable into two corresponding components. This condition is referred to as the *additivity* condition.

An additional important concept of fairness is that of a *CORE* of a cost game. The Core of $< C, Z >$ is defined as follows:

$$\text{Core} < C, Z >= \{a \in R^m : \sum_{i=1}^{m} a_i = Z(C) \text{ and } \sum_{c_i \in S} a_i \leq Z(S) \text{ for all } S \subset C \text{ and } S \neq \emptyset\}$$

This Core condition is known as the *sand-alone* cost test. Its rationale is as follows: "if cooperation among parties is to be voluntary, then the calculus of self-interest dictates that no participant - or group of participants - be charged more than their "stand-alone" (opportunity) costs. Otherwise they would have no incentive to agree to the proposed allocation." (Young 1985, p. 8).

Theorem 1: The Core for the TSP cost game is empty.

Proof: By contradiction. Assume that the Core is not empty. This implies that there exists at least one set of m numbers $\{a_1, a_2, ..., a_m\}$ such that $\sum_{i=1}^{m} a_i = Z(C)$ and $\sum_{c_i \in S} a_i \leq Z(S)$ for all nonempty $S \in 2^C$. It is clear that for all the subsets S such that $|S| = 1, Z(S) = 0$. If we assume that $Z(C) > 0$ then it is immediately clear that the Core of the TSP cost game is empty•

Corollary 1: Given a cost game with $Z(C) > 0$ and for each (singleton) $c_i, i = 1, 2, ..., m \quad Z(\{c_i\}) = 0$, then the Core of such a game is clearly empty.

A principle related to the stand-alone cost test is the *incremental cost test.* It states that no participant should be charged less than the marginal cost of including him.

The *incremental* or *marginal* cost of any set S is defined as $Z(C) - Z(C \setminus S)$, and the incremental cost test requires that the cost allocation $a \in R^m$ satisfy $\sum_{c_i \in S} a_i \geq Z(C) - Z(C \setminus S)$ for all $S \subseteq C$.

The interpretation of the incremental cost test is that if it is violated for some S, then the coalition $C \setminus S$ is subsidizing S.

Since requiring an efficient cost allocation which satisfies the incremental cost test in a cost game is equivalent to a nonempty Core, for the TSP cost game the incremental cost test does not hold in general. One can easily prove it independently of Theorem 1 or accept it as a Corollary of Theorem 1.

Still, the incremental cost of each city in an optimal TSP solution is of interest in the context of cost allocation. By examining the properties of the optimal TSP solution one can obtain the following general result for $m > 3$:

Theorem 2: Given that the distance matrix $\{d(c_i, c_j)\}$ satisfies the triangle inequality, then: $Z(C) \geq \sum_{i=1}^{m} |Z(C) - Z(C \setminus \{c_i\})| = mZ(C) - \sum_{i=1}^{m} Z(C \setminus \{c_i\})$, which can be rewritten as: $\sum_{i=1}^{m} Z(C \setminus \{c_i\}) \geq (m-1)Z(C)$.

Proof: The proof follows an induction argument. For simplicity we assume that the optimal TSP tour follows the cities in order: $c_1 \mapsto c_2 \mapsto c_3 \mapsto ... \mapsto c_m \mapsto c_1$ and

that the distances are symmetric.

First we show that this inequality is true (holds with equality) for m = 3 (i.e., $C = \{c_1, c_2, c_3\}$).

$$Z(C) = d(c_1, c_2) + d(c_2, c_3) + d(c_3, c_4)$$

$Z(C) - Z(C \setminus \{c_1\}) = d(c_1, c_2) + d(c_3, c_1) - d(c_2, c_3)$, $Z(C) - Z(C \setminus \{c_2\}) = d(c_1, c_2) + d(c_2, c_3) - d(c_3, c_1)$, $Z(C) - Z(C \setminus \{c_3\}) = d(c_2, c_3) + d(c_3, c_1) - d(c_1, c_2)$

Adding up the incremental costs for the cities c_1, c_2, and c_3 produces Z(C).

Before getting to the next induction stage we need to prove some intermediate results. But first some notation.

In order to facilitate the presentation denote by $m_i = Z(C) - Z(C \setminus \{c_i\})$ as the inceremental cost of city c_i in the TSP solution for the set C. Denote by m'_j the incremental cost for the city $c_j, j \neq i$ given $C' = C \setminus \{c_i\}$.

Given that the triangle inequality holds for our set C of m cities, the following is true: $Z(C) \geq Z(C \setminus \{c_i\})$, for all $c_i \in C$.

Claim 1: $Max_{c_i \in C}\{\sum_{j=1, j \neq i} m'_j\} \geq \sum_{i=1}^{m-1} m_{\pi(i)}$ for all permutations π of 1,2, ..., m.

Proof: The proof follows the argument that by taking out the city which increases the subsequent incremental costs the most, they add up to at least m - 1 original incremental costs. Note that in general, given a fixed set of cities $C = \{c_1, c_2, ..., c_m\}$, by adding a city which does not maximize the sum of the incremental costs, the corresponding incremental costs of the cities in the set C' decrease.

Returning to our inductive argument in the proof of Theorem 2 we assume that the claim of Theorem 2 is true for all sets $S \subset C$, such that $|S| = m - 1$, then:

$$Z(C \setminus \{c_i\}) \geq \sum_{j=1, j \neq i}^{m} m'_j \quad \text{for all } c_i \in C$$

$$Z(C) \geq Max_{c_i \in C} Z(C \setminus \{c_i\}) \geq Max_{c_i \in C}\{\sum_{j=1, j \neq i}^{m} m'_j\}$$

$$Z(C) - Max_{c_i \in C}\{\sum_{j=1, j \neq i}^{m} m'_j\} \geq Z(C) - Max_{c_i \in C} Z(C \setminus \{c_i\}) = Min_{c_i \in C} m_i$$

Using the inequality from Claim 1 we obtain the following:

$$Z(C) - \sum_{i=1}^{m-1} m_{\pi(i)} \geq Min_{c_i \in C} m_i \quad \text{for all permutations } \pi \text{ of } 1,2, ..., m.$$

Thus, $Z(C) \geq \sum_{i=1}^{m} m_i$ •

Whether Theorem 2 is true for distance matrices for which the triangle inequality does not hold is an open question. We conjucture that the result holds for more general distance matrices for which the monotonicity of the Z(S) function is satisfied. For the general case one can easily devise an TSP instance for which $Z(C) < Z(C \setminus$

$\{c_j\}$) for some city $c_j \in C$. In any case from now on we restrict the presentation to the TSP's for which the triangle inequality is satisfied unless specifically stated to the contrary.

Given that the ideal economic cost allocation - the marginal cost - does not cover the cost of the TSP tour, one has to device alternative cost allocation schemes if the full cost is to be charged to the participants. Since the Core of the TSP cost game is in general empty, we can not preclude the existance of some cost subsidization between parts of the set C in any cost allocation scheme.

3. CANDIDATE ALLOCATION SCHEMES

Next, we examine a number of cost allocation schemes for the TSP.

The simplest cost allocation method is the *egalitarian* method f_E which divides the cost Z(C) equally among the cities in C. I.e., $f_E(< C, Z >) = \frac{Z(C)}{m}$ **e**, where **e** is an m dimensional vector of $1's$. This method is efficient, anonymous, no 'kickbacks', additive, and monotonic. Yet, it is clearly unattractive. It does not relate the cost allocated to city c_i to its marginal cost. In Tijs and Driessen, 1986, they describe cost allocation methods based on separable and nonseparable costs. Those methods consist af two steps; First determine the marginal cost (separable cost) and allocate it to the participants. Next, design a weighting scheme which allocates the remaining (nonseperable) part of the total cost. Tijs and Driessen, 1986, describe a number of such methods considered in the 1930's in connection with the Tennessee Valley Authority project. One such method is the *equal charge method* - (ECM) where after allocating to each participant its marginal cost, the rest is split equally among the participants. We examine this (ECM) cost allocation scheme in some detail.

In Dror and Ferland, 1985, the cost allocation schemes for the TSP is examined with respect to the 'distance' from the marginal cost. More specifically, the cost allocation problem is stated in the following manner:

Given the set $C = \{c_1, c_2, ..., c_m\}$ of cities with their corresponding distance matrix and the marginal costs $m_i, i = 1, 2, ..., m$, solve the following constrained optimization problem:

(CA1) $\qquad MinMax_i\{|x_i - m_i|\}$

subject to:

$$\sum_{i=1}^{m} x_i = Z(C)$$

$$x_i \geq 0, i = 1, 2, ..., m$$

A solution to the (CA1) problem defines a cost allocation scheme which has the least maximal deviation from the marginal cost for the cities in C. An interesting result proven in Dror and Ferland (1985) is that this solution is exactly the ECM

used in the 1930's in the Tennessee Valley Authority project. Moreover, changing the objective function in the (CA1) to Minimizing the $\sum_{i=1}^{m}(x_i - m_i)^{2k}, (k \geq 1)$, does not change the solution, as can be seen from Veinott, 1971.

The ECM cost allocation as related to the TSP is efficient, anonymous, no 'kick-backs', monotonic, and can be viewed as additive. It does not posses the *dummy player property* which can be stated as follows:

Define the *marginal cost contribution relative to S* by:

$$m^i(S) = \begin{cases} Z(S) - Z(S \setminus \{c_i\}) & \text{if } c_i \in S \\ \\ Z(S \cup \{c_i\}) - Z(S) & \text{if } c_i \notin S \end{cases}$$

The city $c_i \in C$ is a **dummy** if c_i contributes nothing to any coalition; i.e., if $m^i(S) = 0$ for all $S \subseteq C$. The *dummy player property* states that if c_i is a dummy in the cost game $< C, Z >$ then the a_i (the cost allocated to c_i) is zero. In the ECM cost allocation, a city can be a dummy but the cost the method allocates to c_i is not zero.

Many other cost allocation methods can be devised for the TSP. We briefly examine two cost allocation methods and comment on their applicability in the case of the TSP.

Tijs and Driessen (1986) examine the properties of what they call the *Cost Gap Allocation Method* - (CGA). They define a cost gap function $g^z : 2^C \mapsto R$ of the cost game $< C, Z >$.

$$g^z(S) = Z(S) - \sum_{c_i \in S} m_i \quad \text{for } S \neq \emptyset \quad g^z(\emptyset) = 0$$

If the cost allocation to participant c_i exceeds the amount $m_i + g^z(S)$ for some coalition $S, (c_i \in S)$, then participant c_i will try to form a coalition S and allocate the cost Z(S) in such a way that for $c_j \in S, j \neq i$ the cost allocation is m_j while c_i absorbes the additional costs in the form of $m_i + g^z(S)$ and still is better off than before. This very attractive cost allocation is based on the assumption that for subsets $S \subset C; Z(S) \geq \sum_{c_i \in S} m_i$ which is clearly not the general case for the TSP.

The other cost allocation method is that of allocating the Shapley value (See Shapley, 1953) to each city c_i. The Shapley value is defined by:

$$Sh_j = \sum_{S \subseteq C, c_j \in S} \frac{(m-|S|)!(|S|-1)!}{m!}[Z(S) - Z(S \setminus \{c_j\})]$$

This value is interpreted as the average marginal cost contribution each city would make to the cost of the optimal TSP tour if it were to form one city at a time optimal tour. The importance of this cost allocation method is in its properties. A cost allocation which is efficient, anonymous, additive, and possesses the dummy player property is the Shapley value cost allocation. It is if and only if characterization.

In the case of the TSP cost game, computing the Z(C) value is already classified as an NP-complete problem (See Garey and Johnson, 1979). The calculation of the Shapley value for the TSP cost game is not even in NP. Thus, regardless of its very attractive properties, the Shapley value does not seem to offer a computationally attainable alternative for TSP cost allocation given a nontrivial size of the TSP.

4. THE TSP WITH A HOME CITY

What happens to the cost allocation question when we reformulate the TSP in the spirit of the nonmathematical description of Hoffman and Wolfe ? I.e., the salesman has a home city !

Mathematically, this reformulation can have the following form: (Addaptation from Bodin et al., 1983.)

Find the variables x_{ij}^{v} for i,j,v = 1,2, ..., m, that minimize

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{v=1}^{m} d(c_i, c_j) x_{ij}^{v} \qquad (6)$$

Subject to:

$$\sum_{i=1}^{m} \sum_{v=1}^{m} x_{ij}^{v} = 1 \qquad j = 2,3, ..., m \qquad (7)$$

$$\sum_{j=1}^{m} \sum_{v=1}^{m} x_{ij}^{v} = 1 \qquad i = 2,3, ..., m \qquad (8)$$

$$\sum_{i=1}^{m} x_{ip}^{v} - \sum_{j=1}^{m} x_{pj}^{v} = 0 \qquad v = 1,2, ..., m;\ p = 1,2, ..., m \qquad (9)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{m} x_{ij}^{v} \leq m - 1 \qquad v = 1,2, ..., m \qquad (10)$$

$$\sum_{i=2}^{m} x_{i1}^{v} \leq 1 \qquad v = 1,2, ..., m \qquad (11)$$

$$\sum_{j=2}^{m} x_{1j}^{v} \leq 1 \qquad v = 1,2, ..., m \qquad (12)$$

$$X \in N \qquad (13)$$

$$x_{ij}^{v} = 0 \text{ or } 1 \text{ for all } i,j,v \qquad (14)$$

The problem is formulated as an m-vehicle routing problem with c_1 designated as the home city and a demand of one unit at each nonhome location. The constraints (10) are so called the capacity constraints and their statement is that no vehicle can visit more than m - 1 cities (other than home). Constraints (13) are the subtour elimination constraints which can be expressed explicitly in many forms (See Bodin et al., 1983.). The triangle inequality of the $\{d(c_i, c_j)\}$ matrix insures that in the optimal solution only one vehicle will be used which corresponds to solving this problem as a TSP. Yet, from the cost allocation point of view the difference between the (1) - (5) formulation and the (6) - (14) formulation is significant.

First, in this definition of the TSP (i.e., adding the concept of the home city), each subset $S \subseteq C$ has to contain the element c_1 and can be represented as $S = \{c_1\} \cup S'$

where $S' \in 2^{C\setminus\{c_1\}}, S' \neq \emptyset$.

It becomes immediately clear that the subadditivity of the functional Z(.) is achived. I.e., for $S', T' \in 2^{C\setminus\{c_1\}}, S', T' \neq \emptyset$, and $S' \cap T' = \emptyset$, $Z(S \cup T) \leq Z(S) + Z(T)$. (Note that this subadditivity of Z does not depend on the choice of c_1.)

Another important question is whether the Core of this modified TSP is empty or not. One strong condition guaranteeing the nonemptiness of the Core is that Z be concave (or submodular) which means that $Z(S \cup T) + Z(S \cap T) \leq Z(S) + Z(T)$ for all $S', T' \in 2^{C\setminus\{c_1\}}$. In Shapley, 1971, it has been shown that given Z concave would imply nonemptiness of the Core. Unfortunatly, concavity condition in general does not hold in this case of the TSP. Thus, the emptiness (or not) of the Core for this modified TSP is not settled in this paper.

One advantage of establishing the subadditivity of Z is that the cost gap function $g^z(S), S = S' \cup \{c_1\}$ is nonnegative for all nonempty $S' \subseteq C \setminus \{c_1\}$. Following Tijs and Driessen 1986, we can now devise a cost gap allocation method - CGA for this modified TSP.

Denote by $w_i = min_{S';c_i \in S'} g^z(S)$ for all $c_i \in C \setminus \{c_1\}$

The number w_i can be viewed as the maximal contribution of the city c_i to the nonseparable cost $g^z(C)$. Tijs and Driessen also assume that $\sum_{i=1}^{m} w_i \geq g^z(C)$. This is not necessary for their method and indeed thought clearly $w_i \geq 0$ for all i (for c_1 set $w_1 = 0$) we do not have the proof for it in the case of the TSP. The cost allocated to $c_i, i = 1, 2, ..., m$ is given according to the CGA by:

$$a_i = m_i + \frac{g^z(C)}{\sum_{i=1}^{m} w_i} w_i \quad \text{i = 1,2, ..., m}$$

(The home city is allocated zero cost since $w_1 = 0$.)

Theorem 3: The cost gap allocation method CGA is efficient - (A1), no 'kick-backs' - (A2), Has the round trip property - (A3) (this property is also called *individually rational* by Tijs and Driessen, 1986), it has the anonymity property - (A4), monotonicity - (A5), and additivity (A6). In the case of the modified TSP it still does not possess the dummy player property.

As we can see, this cost allocation method would probably receive high marks according to Fishburn and Pollak, 1983. The proofs of the claims in the above theorem are stright forward and are omitted in this exposition.

5. THE BIN PACKING COST ALLOCATION

The so called Bin Packing Problem - (BPP) is another well known 'hard' combinatorial optimization problem which received extensive attantion in the professional literature. (See Garey and Johnson, 1979.) The problem's description is very simple:

Given a finite set $U = \{u_1, u_2, ..., u_n\}$ of items each of rational size between zero and one denoted by w_i ; i = 1,2, ..., n, find a partition of U into the smallest number of disjoint subsets $U_1, U_2, ...$, such that the total size of each subset does not exceed one.

The corresponding mathematical formulation of this BPP is:

$$\text{Minimize } Z = \sum_{j=1}^{m} c_j y_j \tag{15}$$

subject to:

$$\sum_{j=1}^{m} x_{ij} = 1 \qquad i = 1,2, ..., n \tag{16}$$

$$\sum_{i=1}^{n} w_i x_{ij} \leq y_j \qquad j = 1,2, ..., m \tag{17}$$

$$x_{ij}, y_j = 0 \text{ or } 1 \qquad i = 1,2, ..., n; j = 1,2, ..., m \tag{18}$$

The binary decision variable x_{ij} takes the value 1 if the item u_i is assigned to bin j and is zero otherwise and the binary decision variable y_j takes the value 1 if the j^{th} bin is used. Otherwise y_j is zero. We express the cost of the j^{th} bin by c_j which in the generic BPP has a value of one for all j and m is taken as a big number to allow a feasible solution to the problem. We assume that the c_j's are positive numbers of about the same magnitude and ordered according to nondecreasing value. I.e., $0 < c_1 \leq c_2 \leq ... \leq c_m$.

An optimal solution for this formulation of the BPP ((16) - (18)) simply minimizes the number of bins used.

Given an optimal solution of the BPP for a given set of items U and the bin costs, how should the total cost be appropriated among the items in U ? In this case, even a simpler question of what constitutes the incremental or marginal cost for the item u_i is not entirely clear.

Denote the optimal value for the BPP by Z(U) given the set U and the bin costs $c_1, c_2, ..., c_m$. The usual way of calculating an incremental cost of item u_i would be $Z(U) - Z(U \setminus \{u_i\})$. This calculation might in many cases produce the value zero for all the items $u_i \in U$. A different way of approximating the incremental cost for item u_i is the following:

Assume that K is the optimal number of bins in the solution. Denote by $M_i \subset U, u_i \in M_i$, the minimal cardinality subset such that when solving the BPP for $U \setminus M_i$ the optimal number of bins decreases by one, (i.e., $Z(U \setminus M_i) = Z(U) - c_K$).

Next, assign the cost $\frac{w_i c_K}{\sum_{u_i \in M_i} w_i}$ denoted as m_i as the incremental cost for the item u_i.

In case the subset M_i is not uniquely defined, break the ties in nondecreasing order of the total size of items in M_i.

Clearly $m_i > 0$ for all $u_i \in U$. Whether $\sum_{i=1}^{n} m_i \leq$ or $> Z(U)$ depends on the

data of each BPP. Can we devise a cost allocation method for the u_i's that is in some sense 'fair' ? An efficient method based on minimizing the maximal deviation from the incremental cost for each u_i leads again to the equal charge method - (ECM). The problem is formulated in the format identical to CA1 and depending on whether $\sum_{i=1}^{n} m_i > Z(U)$ or $\sum_{i=1}^{n} m_i \leq Z(U)$ the ECM would entail that the allocated costs a_i be: $a_i = m_i - \frac{\sum_{i=1}^{n} m_i - Z(U)}{n}$ or $a_i = m_i + \frac{Z(U) - \sum_{i=1}^{n} m_i}{n}$ respectively.

This ECM cost allocation method is efficient, no 'kickbacks', has the anonymity property, is monotonic and additive. The dummy player property seems to be a problem for this BPP cost game as well.

An interesting fact is that the functional Z(.) for the BPP is concave when combined with the incremental cost m_i as defined here. I.e., $S \subseteq S'$ implies that $m_i(S) \geq m_i(S')$. This concavity, according to Shapley, 1971, would imply nonempty Core of Z.

The cost gap function for this cost game is not necessarily nonnegative, thus, the CGA method is of no use. The Shapley value seems inappropriate since $Z(S) - Z(S \setminus \{u_i\})$ for $S \subset U$ might be zero for all S but singletons.

6. THE KNAPSACK COST ALLOCATION

Consider loading a vessel for one leg journey. The cost of this journey is fixed regardless of what the vessel is carying. Denote this cost by C. There is a finite set $U = \{u_1, u_2, ..., u_n\}$ of potential items to load on that vessel. Each item u_i has an associated volume $w_i > 0$ and a revenue of $r_i > 0$. The total 'size' of the vessel is W. The problem faced in this case is simply the selection of items which will be loaded on the vessel for this journey given that $\sum_{i=1}^{n} w_i > W$ so that the total revenue collected will be maximal. Mathematical formulation of this problem is the following:

$$\text{Maximize } Z = \sum_{i=1}^{n} r_i y_i - C \qquad (19)$$

Subject to:

$$\sum_{i=1}^{n} w_i y_i \leq W \qquad (20)$$

$$y_i = 0 \text{ or } 1 \text{ for all i} = 1,2, ..., n \qquad (21)$$

This problem is well known as the 0-1 knapsack problem. The cost of the journey C is irrelevant when solving this optimization problem.

If we are interested in computing the profit (revenue minus cost) contribution of each item that is being shipped on this vessel, we have to assign shipping costs to each of the items selected. This cost allocation problem is very simple if the total $(\sum_{i=1}^{n} w_i y_i)$ volume of items selected for this leg equals exactly W. It would be hard to argue that in this case a 'fair' cost allocation for the items in the solution is not $a_i = C \frac{w_i}{\sum_{i=1}^{n} w_i y_i}$. A more complicated case for this cost allocation problem is the case

where for the optimal solution of the knapsack problem the total volume of the items selected is strictly less than W. In this section we confine the discussion to that case.

Denote the optimal solution vector to the knapsack problem by $Y^* = \{y_1^*, y_2^*, ..., y_n^*\}$ where the y_i^* takes the value 1 if the item u_i is selected for loading and is 0 otherwise. The incremental cost for the items selected is defined as:

$$m_i = Z(U) - Z(U \setminus \{u_i\}) = rY^* - rY_i^*$$

where r is the revenue vector and Y_i^* is the optimal solution vector for the set $U \setminus \{u_i\}$. For the items u_i not in the solution $m_i = 0$.

In general , one can not assume that $\sum_{i=1}^{n} m_i y_i^* \leq C$. It clearly depends on the problem's data. What is the 'appropriate' cost allocation method for this knapsack problem ?

We can implement the ECM cost allocation in the same manner as for the BPP and obtain essentially the same properties as in the BPP case. The CGA method does not seem applicable to the knapsack since the cost gap function is not necessarily nonnegative. A viable alternative cost allocation method in this case seems to be the Shapley value. One can compute with moderate computational effort the Shapley values in the case of knapsack problems of size into the thousands in the number of items. In addition, the marginal cost m_i is monotonically decreasing function of S ($S \subseteq U$) which implies concavity of Z(.) and in turn ensures the nonemptiness of the Core for this knapsack cost game.

Since the knapsack problem is the 'easier' problem in terms of complexity than the TSP and the BPP it seems to offer a wider choice of computationally affordable cost allocation methods. (The knapsack is a an NP-complete problem which can be solved via a pseudo-polynomial algorithm by dynamic programming, Garey and Johnson, 1979.) The method actually adopted depends as usual on the accouting practices and managerial objectives.

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